

A NOTE ON THE LIOUVILLE FUNCTION IN SHORT INTERVALS

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ABSTRACT. In this note we give a short and self-contained proof that, for any $\delta > 0$, $\sum_{x \leq n \leq x+x^\delta} \lambda(n) = o(x^\delta)$ for almost all $x \in [X, 2X]$. We also sketch a proof of a generalization of such a result to general real-valued multiplicative functions. Both results are special cases of results in our more involved and lengthy recent pre-print.

1. INTRODUCTION

In our recent pre-print [4] we have proved (among other things) the following theorem.

Theorem 1. *Let $f : \mathbb{N} \rightarrow [-1, 1]$ be a multiplicative function, and let $h = h(X) \rightarrow \infty$, arbitrarily slowly with $X \rightarrow \infty$. Then, for almost all $X \leq x \leq 2X$,*

$$\frac{1}{h} \sum_{x \leq n \leq x+h} f(n) = \frac{1}{X} \sum_{X \leq n \leq 2X} f(n) + o(1)$$

with $o(1)$ not depending on f .

In particular for the Liouville function this result implies that, for any $\psi(X) \rightarrow \infty$ with $X \rightarrow \infty$, we have

$$(1) \quad \sum_{x \leq n \leq x+\psi(X)} \lambda(n) = o(\psi(X))$$

for almost all $X \leq x \leq 2X$. Previously this was known unconditionally only when $\psi(X) \geq X^{1/6}$ (using zero-density theorems), and under the density hypothesis for $\psi(X) \geq X^\delta$ for any $\delta > 0$.

The proof of Theorem 1 is complicated for two reasons. First of all, in order to achieve the result for a specific function such as $\lambda(n)$ with h growing arbitrarily slowly we need to perform a messy decomposition of the Dirichlet polynomial

$$\sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}}$$

according to the size of $\sum_{P < p < Q} \lambda(p)p^{-1-it}$ for suitable intervals $[P, Q]$. Secondly, to obtain the result for arbitrary f , we need to input some additional ideas dealing with large values of Dirichlet polynomials. We realized recently that in the special case of the Liouville function and intervals of length X^δ neither is necessary.

Our goal in this short note is to give a short and self-contained proof of the following special case of Theorem 1.

Theorem 2. *Let $\delta > 0$ be given. Then, for almost all $X \leq x \leq 2X$, we have*

$$\sum_{x \leq n \leq x+X^\delta} \lambda(n) = o(X^\delta).$$

We have not tried to optimize any of the bounds for the amount of cancellations or for the size of the exceptional set. With a bit additional effort this can be done (but we refer the reader to our paper [4]).

For the convenience of the reader we have also indicated in the appendix how to generalize this result to arbitrary multiplicative f . This is more intricate and depends on a number of lemmas which are proven in our paper [4]. We will invoke these lemmas freely throughout the proof of the following theorem.

Theorem 3. *Let $f : \mathbb{N} \rightarrow [-1, 1]$ be a multiplicative function. Let $\delta > 0$. Then, for almost all $X \leq x \leq 2X$, we have*

$$\frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} f(n) = \frac{1}{X} \sum_{X \leq n \leq 2X} f(n) + o(1)$$

with $o(1)$ not depending on f .

It is worthwhile to point out that essentially the only non-standard idea from [4] that is needed in the proof of Theorem 2 is the use of Ramaré type identity (see (3) below).

2. THE MAIN PROPOSITIONS

Theorem 2 follows immediately from the following proposition.

Proposition 1. *Let $\delta > 0$ be given. Then, for any $\varepsilon > 0$,*

$$\int_X^{2X} \left| \frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} \lambda(n) \right|^2 dx \ll_\varepsilon \frac{X}{(\log X)^{1/3-\varepsilon}}.$$

Deduction of Theorem 2 from Proposition 1. By Chebyshev's inequality the number of exceptional $x \in [X, 2X]$ for which

$$\left| \frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} \lambda(n) \right| \geq \frac{1}{(\log X)^{1/9}}.$$

is at most

$$(\log X)^{2/9} \int_X^{2X} \left| \frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} \lambda(n) \right|^2 dx \ll_\varepsilon \frac{X}{(\log X)^{1/9-\varepsilon}} = o(X)$$

as claimed. \square

In order to prove Theorem 3 we will sketch the proof of the following proposition in the Appendix.

Proposition 2. *Let $f : \mathbb{N} \rightarrow [-1, 1]$ be a multiplicative function. Let $\delta > 0$ be given. Then*

$$\int_X^{2X} \left| \frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} f(n) - \frac{1}{X} \sum_{X \leq n \leq 2X} f(n) \right|^2 dx \ll \frac{X}{(\log X)^{1/48}}.$$

3. LEMMAS

In Lemma 4 below we relate the integral in Proposition 1 to a mean square of a Dirichlet polynomial. To deal with this, we use the following three standard lemmas.

Lemma 1. *Let $A > 0$ be given. We have, uniformly in $|t| \leq (\log X)^A$,*

$$\sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \ll (\log X)^{-A}.$$

Proof. By the prime number theorem for any $A > 0$, we have,

$$\sum_{X \leq n \leq u} \frac{\lambda(n)}{n} \ll (\log X)^{-2A}$$

for any $u \in [X, 2X]$. Therefore, integrating by parts we find

$$\begin{aligned} \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} &= \int_X^{2X} u^{-it} d \sum_{X \leq n \leq u} \frac{\lambda(n)}{n} \\ &\ll \frac{|t|}{X} \int_X^{2X} \left| \sum_{X \leq n \leq u} \frac{\lambda(n)}{n} \right| du + (\log X)^{-2A} \\ &\ll (\log X)^A \cdot (\log X)^{-2A} = (\log X)^{-A} \end{aligned}$$

which gives the claim. \square

Lemma 2. *Let $A > 0$ be given and $X \geq 1$. Assume that $\exp((\log X)^\theta) \leq P \leq Q \leq X$ for some $\theta > 2/3$ and let*

$$\mathcal{P}(1+it) = \sum_{P \leq p \leq Q} \frac{1}{p^{1+it}}.$$

Then, for any $|t| \leq X$,

$$|\mathcal{P}(1+it)| \ll \frac{\log X}{1+|t|} + (\log X)^{-A}.$$

Proof. In case $|t| \leq 10$, the claim follows immediately from the prime number theorem, so we can assume $|t| > 10$. We can also assume that fractional parts of P and Q are $1/2$ each. Perron's formula says that, for any $\kappa > 0$ and $y > 0$, we have

$$\frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} y^s \cdot \frac{ds}{s} = \begin{cases} 1 & \text{if } y > 1 \\ 0 & \text{if } y < 1 \end{cases} + O\left(\frac{y^\kappa}{\max(1, T|\log y|)}\right).$$

Therefore, letting $\kappa = 1/\log X$, and $T = (|t|+1)/2 < |t|-1$, we have

$$(2) \quad \mathcal{P}(1+it) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \log \zeta(s+1+it) \cdot \frac{Q^s - P^s}{s} \cdot ds + O\left(\frac{\log X}{|t|+1} + \frac{1}{P^{1/2}}\right).$$

Using Vinogradov's zero-free region, we see that $\log \zeta(s+1+it)$ is well defined in the region

$$\mathcal{R} : 1 \leq |\Im s + t| \leq 2X, \quad \Re s \geq -\sigma_0 := -\frac{1}{(\log X)^{2/3}(\log \log X)}.$$

In addition for $s \in \mathcal{R}$ we have $|\log \zeta(s+1+it)| \ll (\log X)^2$. Therefore shifting the contour in (2) to the edge of this region, we see that

$$\begin{aligned} \mathcal{P}(1+it) &= \frac{1}{2\pi i} \int_{-T}^T \log \zeta(1-\sigma_0+iu+it) \cdot \frac{Q^{-\sigma_0+iu} - P^{-\sigma_0+iu}}{-\sigma_0+iu} du + O\left(\frac{\log X}{|t|+1} + \frac{1}{P^{1/2}}\right) \\ &\ll (\log X)^{-A} + \frac{\log X}{|t|+1}. \end{aligned}$$

as claimed. \square

Lemma 3. *One has*

$$\int_{-T}^T \left| \sum_{n \sim X} \frac{a_n}{n^{1+it}} \right|^2 \ll (T+X) \sum_{n \sim X} \frac{|a_n|^2}{n^2}.$$

Proof. See [3, Theorem 9.1]. \square

4. PROOF OF PROPOSITION 1

We start with the following lemma which is in the spirit of previous work on primes in almost all intervals, see for instance [2, Lemma 9.3].

Lemma 4. *Let $\delta > 0$ be given. Then*

$$\begin{aligned} & \frac{1}{X} \int_X^{2X} \left| \frac{1}{X^\delta} \sum_{\substack{x \leq n \leq x+X^\delta}} \lambda(n) \right|^2 dx \\ & \ll \int_0^{X^{1-\delta}} \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt + \max_{T > X^{1-\delta}} \frac{X^{1-\delta}}{T} \int_T^{2T} \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt. \end{aligned}$$

Proof. Write $h := X^\delta$. By Perron's formula

$$\frac{1}{h} \sum_{x \leq n \leq x+h} \lambda(n) = \frac{1}{h} \cdot \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \left(\sum_{n \sim X} \frac{\lambda(n)}{n^s} \right) \cdot \frac{(x+h)^s - x^s}{s} ds.$$

Hence it is enough to bound

$$V := \frac{1}{h^2 X} \int_X^{2X} \left| \int_1^{1+i\infty} F(s) \frac{(x+h)^s - x^s}{s} ds \right|^2 dx,$$

where $F(s) = \sum_{n \sim X} \lambda(n) n^{-s}$. We would like to add a smoothing, take out a factor x^s , expand the square, exchange the order of integration and integrate over x . However, the term $(x+h)^s$ prevents us from doing this and we overcome this problem in a similar way to [5, Page 25]. We write

$$\begin{aligned} \frac{(x+h)^s - x^s}{s} &= \frac{1}{2h} \left(\int_h^{3h} \frac{(x+w)^s - x^s}{s} dw - \int_h^{3h} \frac{(x+w)^s - (x+h)^s}{s} dw \right) \\ &= \frac{x}{2h} \int_{h/x}^{3h/x} x^s \frac{(1+u)^s - 1}{s} du - \frac{x+h}{2h} \int_0^{2h/(x+h)} (x+h)^s \frac{(1+u)^s - 1}{s} du. \end{aligned}$$

where we have substituted $w = x \cdot u$ in the first integral and $w = h + (x+h)u$ in the second integral. Let us only study the first summand, the second one being handled completely similarly. Thus we assume that

$$\begin{aligned} V &\ll \frac{X}{h^4} \int_X^{2X} \left| \int_{h/x}^{3h/x} \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds du \right|^2 dx \\ &\ll \frac{1}{h^3} \int_{h/(2X)}^{3h/X} \int_X^{2X} \left| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx du \\ &\ll \frac{1}{h^2 X} \int_X^{2X} \left| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx \end{aligned}$$

for some $u \ll h/X$.

Let us introduce a smooth function $g(x)$ supported on $[1/2, 4]$ and equal to 1 on $[1, 2]$. We obtain

$$\begin{aligned}
V &\ll \frac{1}{h^2 X} \int g\left(\frac{x}{X}\right) \left| \int_1^{1+i\infty} F(s) x^s \frac{(1+u)^s - 1}{s} ds \right|^2 dx \\
&\leq \frac{1}{h^2 X} \int_1^{1+i\infty} \int_1^{1+i\infty} \left| F(s_1) F(s_2) \frac{(1+u)^{s_1} - 1}{s_1} \frac{(1+u)^{s_2} - 1}{s_2} \right| \left| \int g\left(\frac{x}{X}\right) x^{s_1 + \overline{s_2}} dx \right| |ds_1 ds_2| \\
&\ll \frac{1}{h^2 X} \int_1^{1+i\infty} \int_1^{1+i\infty} |F(s_1) F(s_2)| \min\left\{\frac{h}{X}, \frac{1}{|t_1|}\right\} \min\left\{\frac{h}{X}, \frac{1}{|t_2|}\right\} \frac{X^3}{|t_1 - t_2|^2 + 1} |ds_1 ds_2| \\
&\ll \frac{X^2}{h^2} \int_1^{1+i\infty} \int_1^{1+i\infty} \frac{|F(s_1)|^2 \min\{(h/X)^2, |t_1|^{-2}\} + |F(s_2)|^2 \min\{(h/X)^2, |t_2|^{-2}\}}{|t_1 - t_2|^2 + 1} |ds_1 ds_2| \\
&\ll \int_1^{1+iX/h} |F(s)|^2 |ds| + \frac{X^2}{h^2} \int_{1+iX/h}^{1+i\infty} \frac{|F(s)|^2}{|t|^2} |ds|.
\end{aligned}$$

The second summand is

$$\ll \frac{X^2}{h^2} \int_{1+iX/(2h)}^{1+i\infty} \frac{1}{T^3} \int_{1+iT}^{1+i2T} |F(s)|^2 |ds| dT \ll \frac{X^2}{h^2} \cdot \frac{1}{X/h} \max_{T \geq X/(2h)} \frac{1}{T} \int_{1+iT}^{1+i2T} |F(s)|^2 |ds|$$

and the claim follows. \square

Proposition 1 will follow from combining Lemma 4 with the following lemma.

Lemma 5. *Let $\delta > 0$ be given. Then,*

$$\int_0^T \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt \ll_\varepsilon \frac{1}{(\log X)^{1/3-\varepsilon}} \cdot \left(\frac{T}{X} + 1 \right) + \frac{T}{X^{1-\delta/2}}.$$

Proof. Since the mean value theorem (Lemma 3) gives the bound $O(\frac{T}{X} + 1)$, we can assume $T \leq X$. Furthermore, by Lemma 1, the part of the integral with $t \leq T_0 := (\log X)^{10}$ contributes $O((\log X)^{-10})$.

Let us now concentrate to the integral over $[T_0, T]$ with $T \leq X$. Let $P = \exp((\log X)^{2/3+\varepsilon})$ and $Q = X^{\delta/3}$. We use the decomposition

$$\begin{aligned}
(3) \quad \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} &= \sum_{P \leq p \leq Q} \frac{\lambda(p)}{p^{1+it}} \sum_{m \sim X/p} \frac{\lambda(m)}{(\#\{p \in [P, Q] : p \mid m\} + 1)m^{1+it}} + \sum_{\substack{n \sim X \\ p \mid n \implies p \notin [P, Q]}} \frac{\lambda(n)}{n^{1+it}},
\end{aligned}$$

which is a variant of Ramaré's identity [1, Section 17.3]. Writing $a_m = \lambda(m)/(\#\{p \in [P, Q] : p \mid m\} + 1)$, we obtain

$$(4) \quad \begin{aligned} \int_{T_0}^T \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt &\ll \int_{T_0}^T \left| \sum_{P \leq p \leq Q} \frac{1}{p^{1+it}} \sum_{m \sim X/p} \frac{a_m}{m^{1+it}} \right|^2 dt + \\ &+ \int_{T_0}^T \left| \sum_{\substack{n \sim X \\ p \mid n \implies p \notin [P, Q]}} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt. \end{aligned}$$

We estimate the second term by completing the integral to $|t| \leq T$ and by applying the mean-value theorem (Lemma 3). This shows that the second term is bounded by

$$\ll (T + X) \frac{1}{X^2} \sum_{\substack{n \sim X \\ p \mid n \implies p \notin [P, Q]}} 1 \ll \left(\frac{T}{X} + 1 \right) \cdot \frac{\log P}{\log Q} \ll \frac{1}{(\log X)^{1/3-\varepsilon}} \cdot \left(\frac{T}{X} + 1 \right)$$

by the fundamental lemma of the sieve. To deal with the first term in (4), we would like to dispose of the condition $mp \sim x$, so that we can use lemmas in Section 3 to Dirichlet polynomials over p and m separately. To do this, we let $H = (\log X)^5$ and split the summations in the appearing Dirichlet polynomial into short ranges, getting

$$(5) \quad \sum_{P \leq p \leq Q} \frac{1}{p^s} \sum_{m \sim X/p} \frac{a_m}{m^s} = \sum_{\lfloor H \log P \rfloor \leq j \leq H \log Q} \sum_{\substack{e^{j/H} \leq p \leq e^{(j+1)/H} \\ P \leq p \leq Q}} \frac{1}{p^s} \sum_{\substack{X e^{-(j+1)/H} \leq m \leq 2X e^{-j/H} \\ X \leq mp \leq 2X}} \frac{a_m}{m^s}.$$

Now we can remove the condition $X \leq mp \leq 2X$ over-counting at most by the integers mp in the ranges $[X e^{-1/H}, X]$ and $[2X, 2X e^{1/H}]$. Therefore we can, for some bounded d_m , rewrite (5) as

$$\sum_{\lfloor H \log P \rfloor \leq j \leq H \log Q} Q_{j,H}(s) F_{j,H}(s) + \sum_{X e^{-1/H} \leq m \leq X} \frac{d_m}{m^s} + \sum_{2X \leq m \leq 2X e^{1/H}} \frac{d_m}{m^s}$$

where

$$Q_{j,H}(s) := \sum_{e^{j/H} \leq p \leq e^{(j+1)/H}} \frac{1}{p^s} \quad \text{and} \quad F_{j,H}(s) := \sum_{X e^{-(j+1)/H} \leq m \leq 2X e^{-j/H}} \frac{a_m}{m^s}.$$

Using this decomposition, applying Cauchy-Schwarz and then taking the maximal term in the resulting sum, we get

$$\begin{aligned} \int_{T_0}^T \left| \sum_{P \leq p \leq Q} \frac{1}{p^{1+it}} \sum_{m \sim X/p} \frac{a_m}{m^{1+it}} \right|^2 dt &\ll (H \log(Q/P))^2 \int_{T_0}^T \left| Q_{j,H}(1+it) F_{j,H}(1+it) \right|^2 dt + \\ &+ \int_{T_0}^T \left| \sum_{Xe^{-1/H} \leq m \leq X} \frac{d_m}{m^{1+it}} \right|^2 dt + \int_{T_0}^T \left| \sum_{2X \leq m \leq 2Xe^{1/H}} \frac{d_m}{m^{1+it}} \right|^2 dt. \end{aligned}$$

for some $j \in [\lfloor H \log P \rfloor, H \log Q]$ depending at most on X and T . We compute the last two integrals by completing the integral to $|t| \leq T$, and applying the mean value theorem (Lemma 3). This way we see that they are bounded by

$$\ll (T+X) \frac{1}{X^2} \cdot (Xe^{1/H} - X) \ll \left(\frac{T}{X} + 1 \right) \frac{1}{H} = \frac{1}{(\log X)^5} \left(\frac{T}{X} + 1 \right).$$

Finally, since $X^{\delta/3} = Q \geq e^{j/H} \geq P/e > \exp((\log X)^{2/3+\varepsilon/2})$, using Lemma 2 we have, for $T_0 \leq t \leq X$,

$$|Q_{j,H}(1+it)| \ll (\log X)^{-9}.$$

Therefore, by the mean value theorem (Lemma 3),

$$\begin{aligned} \int_{T_0}^T |Q_{j,H}(1+it) F_{j,H}(1+it)|^2 dt &\ll (\log X)^{-18} \int_{T_0}^T |F_{j,H}(1+it)|^2 dt \\ &\ll (\log X)^{-18} \cdot (T + Xe^{-j/H}) \frac{1}{Xe^{-j/H}} \\ &\ll (\log X)^{-18} \cdot \left(\frac{QT}{X} + 1 \right) \ll \frac{T}{X^{1-\delta/3}} + \frac{1}{(\log X)^{18}} \end{aligned}$$

since $e^{j/H} \leq Q = X^{\delta/3}$. Combining everything together we get the following bound

$$\int_0^T \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}} \cdot \left(\frac{T}{X} + 1 \right) + (\log X)^{12} \cdot \left(\frac{T}{X^{1-\delta/3}} + \frac{1}{(\log X)^{18}} \right)$$

which implies the required result. \square

We are now ready to prove Proposition 1.

Proof of Proposition 1. Using Lemma 5 we get

$$\int_0^{X^{1-\delta}} \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}}$$

and similarly

$$\max_{T > X^{1-\delta}} \frac{X^{1-\delta}}{T} \int_T^{2T} \left| \sum_{n \sim X} \frac{\lambda(n)}{n^{1+it}} \right|^2 dt \ll \frac{1}{(\log X)^{1/3-\varepsilon}}.$$

We conclude therefore using Lemma 4, that,

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{X^\delta} \sum_{x \leq n \leq x+X^\delta} \lambda(n) \right|^2 dx \ll \frac{1}{(\log X)^{1/3-\varepsilon}}$$

as claimed. \square

5. APPENDIX: PROOF OF PROPOSITION 2

The proof of Proposition 2 is more involved and involves more tools. We will therefore freely make appeal to [4] whenever necessary. First, [4, Lemma 14] (a variant of Lemma 4 here), implies that in order to establish Proposition 2, we need to bound

$$\int_{(\log X)^{1/15}}^T \left| \sum_{n \sim X} \frac{f(n)}{n^{1+it}} \right|^2 dt.$$

and perform a minor cosmetic operation. The main ingredient in the proof of Proposition 2 is thus the following lemma.

Lemma 6. *We have,*

$$\int_{(\log X)^{1/15}}^T \left| \sum_{n \sim X} \frac{f(n)}{n^{1+it}} \right|^2 dt \ll \frac{1}{(\log X)^{1/48}} \cdot \left(\frac{T}{X} + 1 \right) + \frac{TX^{o(1)}}{X}.$$

Proof. In view of the trivial bound $O(T/X+1)$ from the mean value theorem (Lemma 3) we can assume that $T \leq X$.

Let

$$H = (\log X)^{1/48}, \quad P = \exp((\log X)^{1-1/48}), \quad Q = \exp(\log X / \log \log X),$$

and let

$$Q_{j,H}(s) := \sum_{e^{j/H} \leq p \leq e^{(j+1)/H}} \frac{f(p)}{p^s} \quad \text{and} \quad F_{j,H}(s) := \sum_{Xe^{-(j+1)/H} \leq m \leq 2Xe^{-j/H}} \frac{f(m)}{m^s}.$$

Then using [4, Lemma 12] (which is a slightly more involved version of some of the arguments in proof of Lemma 5) we find the following bound,

$$\begin{aligned} & \int_{(\log X)^{1/15}}^T \left| \sum_{n \sim X} \frac{f(n)}{n^{1+it}} \right|^2 dt \ll \\ & \ll (\log X)^{2+1/24} \int_{(\log X)^{1/15}}^T |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 dt + \frac{1}{(\log X)^{1/48}} \cdot \left(\frac{T}{X} + 1 \right) \end{aligned}$$

for some $\lfloor H \log P \rfloor \leq j \leq H \log Q$ depending at most on T and X .

Let us define

$$\begin{aligned} \mathcal{T}_S &= \{t \in [(\log X)^{1/15}, T] : |Q_{j,H}(1+it)| \leq (\log X)^{-100}\} \\ \text{and } \mathcal{T}_L &= \{t \in [(\log X)^{1/15}, T] : |Q_{j,H}(1+it)| > (\log X)^{-100}\}. \end{aligned}$$

On \mathcal{T}_S we have by definition and the mean value theorem (Lemma 3)

$$\begin{aligned} \int_{\mathcal{T}_S} |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 dt &\ll (\log X)^{-200} \int_0^T |F_{j,H}(1+it)|^2 dt \\ &\ll (\log X)^{-200} \cdot (T + X e^{-j/H}) \cdot \frac{1}{X e^{-j/H}} \\ &\ll (\log X)^{-200} \cdot \left(\frac{TX^{o(1)}}{X} + 1 \right) \end{aligned}$$

since $e^{j/H} \leq Q = X^{o(1)}$, which is a sufficient saving in the logarithm since we need to beat $(\log X)^{2+1/24}$ by at least $(\log X)^{1/48}$.

Let us now turn to \mathcal{T}_L . We can find a well-spaced subset $\mathcal{T} \subseteq \mathcal{T}_L$ such that

$$\int_{\mathcal{T}_L} |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 dt \ll \sum_{t \in \mathcal{T}} |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 dt$$

Using [4, Lemma 8], we see that

$$\begin{aligned} |\mathcal{T}| &\ll \exp \left(2 \frac{\log(\log X)^{100}}{j/H} \log T + 2 \log(\log X)^{100} + 2 \frac{\log T}{j/H} \log \log T \right) \\ &\ll \exp \left(\frac{(\log X)^{1+o(1)}}{\log P} \right) \ll \exp((\log X)^{1/48+o(1)}). \end{aligned}$$

In addition, using [4, Lemma 3] (a consequence of Halász's theorem), we find that

$$\sup_{(\log X)^{1/15} \leq |t| \leq T} |F_{j,H}(1+it)| \ll (\log X)^{-1/16} \cdot \frac{\log Q}{\log P} \ll (\log X)^{-1/24}.$$

Therefore using [4, Lemma 11] (a large value result for Dirichlet polynomials over primes) this time, we get

$$\begin{aligned} \sum_{t \in \mathcal{T}} |Q_{j,H}(1+it)F_{j,H}(1+it)|^2 &\ll (\log X)^{-1/12} \sum_{t \in \mathcal{T}} |Q_{j,H}(1+it)|^2 \\ &\ll (\log X)^{-1/12} \cdot \left(e^{j/H} + |\mathcal{T}|e^{j/H} \exp(-(\log X)^{1/5}) \right) \sum_{e^{j/H} < p < e^{(j+1)/H}} \frac{1}{p^2 \log p} \\ &\ll (\log X)^{-1/12} \cdot \frac{e^{(j+1)/H} - e^{j/H}}{e^{j/H}(\log e^{j/H})^2} \ll \frac{1}{H(\log X)^{1/12}(\log P)^2} \ll (\log X)^{-2-1/16}. \end{aligned}$$

Therefore combining everything together we get

$$\int_{(\log X)^{1/15}}^T \left| \sum_{n \sim X} \frac{f(n)}{n^{1+it}} \right|^2 dt \ll (\log X)^{-100} \left(\frac{TX^{o(1)}}{X} + 1 \right) + (\log X)^{-1/48} \left(\frac{T}{X} + 1 \right),$$

and the claim follows. \square

We are finally ready to prove the Proposition.

Proof of Proposition 1. Now, by [4, Lemma 14] and Lemma 6, we get, for $X^\delta = h_1 \leq h_2 = X/(\log X)^{1/5}$,

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h_1} \sum_{x \leq m \leq x+h_1} f(m) - \frac{1}{h_2} \sum_{x \leq m \leq x+h_2} f(m) \right|^2 dx \ll (\log X)^{-1/48}.$$

The claim follows since by [4, Lemma 4]

$$\frac{1}{h_2} \sum_{x \leq n \leq x+h_2} f(n) = \frac{1}{x} \sum_{X \leq n \leq 2X} f(n) + O((\log X)^{-1/20}).$$

\square

REFERENCES

- [1] J. Friedlander and H. Iwaniec. *Opera de cribro*, volume 57 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.
- [2] G. Harman. *Prime-detecting Sieves*, volume 33 of *London Mathematical Society Monographs (New Series)*. Princeton University Press, Princeton, 2007.
- [3] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 2004.
- [4] K. Matomäki and M. Radziwiłł. Multiplicative functions in short intervals. *Preprint*, available at <http://arxiv.org> as arXiv:1501.04585 [math.NT].
- [5] B. Saffari and R. C. Vaughan. On the fractional parts of x/n and related sequences II. *Ann. Inst. Fourier*, 27:1–30, 1977.

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